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1985 J. Phys. A: Math. Gen. 18 L123

(http://iopscience.iop.org/0305-4470/18/3/005)

J. Phys. A: Math. Gen. 18 (1985) L123-L128. Printed in Great Britain

LETTER TO THE EDITOR

Towards a satisfactory formulation of the quantum Langevin equation

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Received 21 November 1984

Abstract. The quantum mechanical Langevin equation for a damped harmonic oscillator with operator-valued noise, $\mathrm{d}A_t/\mathrm{d}t = (-\mathrm{i}\omega - \gamma)A_t + \lambda w(t)$, is reinvestigated. By assuming the canonical commutation relation at initial time, $[A_0, A_0^*] = 1$, and a Hermitian operator for the Gaussian quantum noise $w(t)$, it is shown that Streater's procedure to satisfy $[A_t, A_t^*] = 1$ for all later time yields an improved result: the noise operator $w(t)$ can be determined universally and normalised such that, by choosing $\lambda^2 = 2\gamma(\beta\hbar\omega)^{-1}$, the (symmetrised) power spectrum of $w(t)$ is equal to the physically significant form $\frac{1}{2}\beta\hbar\omega \coth(\frac{1}{2}\beta\hbar\omega)$, where β is the inverse temperature of the heat bath, thus ensuring the β-KMS correlation functions for any pair of operators.

The theoretical formulation of quantum stochastic processes is one of the subjects currently studied in mathematical physics (Lewis 1981, Hudson and Streater 1981, Barnet *et al* 1983a, b, Hudson and Parthasarathy 1984, Nakazawa 1984). On the side of statistical physics there exists an outstanding problem, which can hopefully be resolved in the course of this development: it was raised first by Senitzky (1960) and later examined by Lax (1966), Haken (1970) and Kubo (1969). Namely, we wish to quantise the Langevin equation for a damped harmonic oscillator with frequency ω and the damping constant γ, which may be expressed as

$$\mathrm{d}A_t/\mathrm{d}t = (-\mathrm{i}\omega - \gamma)A_t + \lambda_a(t) \tag{1}$$

$$\mathrm{d}A_t^*/\mathrm{d}t = (\mathrm{i}\omega - \gamma)A_t^* + \lambda a^*(t) \qquad \omega, \gamma > 0, \ \lambda \text{ real} \tag{1*}$$

where A_t (or its complex conjugate A_t^*) represents, for example, the complex amplitude of a mode of the electromagnetic field in a cavity. The well known physical situation behind this set of equations is that the macroscopic decaying behaviour of the oscillator with rate γ is caused by the fluctuating force $\lambda a(t)$ (or $\lambda a^*(t)$) randomly acting on it from the cavity that plays the role of a heat bath whose only property, to the best of our knowledge, is the temperature β^{-1}. The classical theory of Brownian motion tells us that the idealised model of the time correlation between the forces $a(t)$ and $a^*(t')$ is sufficient to describe the oscillator, i.e.

$$\begin{aligned} &\langle a(t)a(t')\rangle = \langle a^*(t)a^*(t')\rangle = 0, \\ &\langle a^*(t)a(t')\rangle = \delta(t-t') \end{aligned} \tag{2}$$

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and then the choice of the noise strength λ can be fixed as

$$\lambda^2 = 2\gamma\langle A^* A\rangle_{\text{equilibrium}} = 2\gamma\beta^{-1} \tag{3}$$

so that every statistical property of the oscillator (e.g. an approach to the thermal equilibrium (Hasegawa and Ezawa 1980)) may be achieved.

To extend the above feature of the classical theory to a quantum framework, Senitzky (1960) and Lax (1966) recognised the necessity of considering the commutation relation between the operators A_t and A_t^*, i.e.

$$\text{(i)} \qquad [A_t, A_t^*] = 1 \quad \text{holds for all time } t \geqslant 0, \tag{4}$$

and that for this it is necessary to regard the noise forces $a(t)$ and $a^*(t)$ also as non-commuting operators. An even more important comment on such noise operators was given by Kubo (1969), who noted that under a stationary situation the time-analyticity property, now known as the KMS condition, should hold:

$$\text{(ii)} \qquad \langle A_t^* A_{t+\tau}\rangle = \langle A_{t+\tau+\mathrm{i}\hbar\beta} A_t^*\rangle. \tag{5}$$

Thereby, he emphasised that the quantum noise fulfilling this condition could not be of the white character in the usual sense. The seemingly satisfactory model of the quantum noise set up by Lax (1966) in a classical white-noise analogue, as a matter of fact, did not satisfy (ii), though it satisfied (i). Thus, the whole question concerning the quantum mechanical version of the Langevin equation (1) can be specified to ask whether it is possible or not to determine the quantum noise $a(t)$ in (1) and $a^*(t)$ in (1*) such that both conditions (i) and (ii) are fulfilled for every pair of solutions A_t and A_t^*.

Notable progress has been made by Streater (1982) along the lines of the above setting. He expanded $a(t)$ and $a^*(t)$ as

$$\begin{aligned} a(t) &= \int_0^\infty \mathrm{d}k\, f(k) a(k)\, \mathrm{e}^{-\mathrm{i}kt}, \\ a^*(t) &= \int_0^\infty \mathrm{d}k\, f^*(k) a^*(k)\, \mathrm{e}^{\mathrm{i}kt} \end{aligned} \tag{6}$$

where

$$[a(k), a^*(k')] = \delta(k-k') \tag{6a}$$

and

$$[a(k), a(k')] = [a^*(k), a^*(k)'] = 0, \tag{6b}$$

and analysed the canonical commutation relation (CCR) (i) to determine $f(k)$, showing that (i) holds if and only if the spectral density function of the oscillators for the bath

$$\rho(k) \equiv \lambda^2 |f(k)|^2 \geqslant 0 \tag{7}$$

satisfies a functional equation

$$\rho(k+\omega) + \rho(-k+\omega) = 2\gamma/\pi. \tag{8}$$

Streater emphasised the positive-axis Fourier expansion (6) so that (8) must be solved by imposing the extra condition

$$\rho(k) = 0 \qquad \text{for } k < 0. \tag{9}$$

This leads to a rather peculiar feature for the possible form of the density function. his conclusion about $\rho(k)$ may be expressed in terms of an arbitrary measurable function $\sigma(k)$; $0 \leqslant \sigma(k) \leqslant 2\gamma/\pi$ as $\rho(k)=0$ $(k<0)$, $=\sigma(\omega-k)(0 \leqslant k<\omega)$, $=2\gamma/\pi-\sigma(k-\omega)$ $(\omega \leqslant k<2\omega)$ and $=2\gamma/\pi$ $(2\omega \leqslant k)$.

With the spectral density function $\rho(k)$ so obtained, Streater further discussed the satisfaction of the condition (ii): for this it is sufficient to assume

$$\begin{aligned} &\langle a^*(k)a(k')\rangle_\beta = n(k)\delta(k-k'), \\ &\langle a(k')a^*(k)\rangle_\beta = (1+n(k))\delta(k-k'), \\ &\langle a(k)a(k')\rangle_\beta = \langle a^*(k)a^*(k')\rangle_\beta = 0 \end{aligned} \tag{10}$$

with the Planck distribution

$$n(k) = (\mathrm{e}^{\beta\hbar k}-1)^{-1} \qquad \text{and} \qquad n(k)+1 = n(k)\,\mathrm{e}^{\beta\hbar k}. \tag{11}$$

One then obtains for the stationary process ($t \gg \gamma^{-1}$ for which the transient component of the solution is dropped)

$$\begin{aligned} \langle A_\infty^* A_\infty(\tau)\rangle_\beta &= \int_0^\infty \mathrm{d}k\,\rho(k)n(k)\frac{\mathrm{e}^{-\mathrm{i}k\tau}}{(k-\omega)^2+\gamma^2} \\ \langle A_\infty(\tau) A_\infty^*\rangle_\beta &= \int_0^\infty \mathrm{d}k\,\rho(k)(1+n(k))\frac{\mathrm{e}^{-\mathrm{i}k\tau}}{(k-\omega)^2+\gamma^2} \\ &= \langle A_\infty^* A_\infty(\tau+\mathrm{i}\hbar\beta)\rangle_\beta \end{aligned} \tag{12}$$

which implies the KMS identity. We note that the satisfaction of this relation stems from the equilibrium characteristics of the heat bath (10) with temperature β^{-1} and holds irrespective of any specific form of the density function $\rho(k)$.

In spite of the detailed construction and internal consistency in the above argument, we feel that some physically unsatisfactory feature remains in that the density function $\rho(k)$ to solve (8) and (9) depends artificially on the frequency ω of the injected oscillator, and even more, on the arbitrary function $\sigma(k)$ which is unknown. So, our discussion in the remainder of this letter will be simply to remedy this deficiency.

Let us consider the quantum noise to be of *Hermitian* character $w(t) \equiv a(t)+a^*(t)$, instead of $a(t)$ or $a^*(t)$ defined by (6), for the Langevin force in (1) or (1*):

$$\mathrm{d}A_t/\mathrm{d}t = (-\mathrm{i}\omega-\gamma)A_t + \lambda w(t) \tag{13}$$

$$\mathrm{d}A_t^*/\mathrm{d}t = (\mathrm{i}\omega-\gamma)A_t^* + \lambda w(t) \tag{13*}$$

where, from (6) and (6*a*, *b*),

$$w(t) = \int_0^\infty \mathrm{d}k f(k)a(k)\,\mathrm{e}^{-\mathrm{i}kt} + \int_0^\infty \mathrm{d}k f^*(k)a^*(k)\,\mathrm{e}^{\mathrm{i}kt}. \tag{14}$$

This choice satisfies the commutation relation

$$\begin{aligned} [w(s), w(s')] &= \int_0^\infty \mathrm{d}k\,|f(k)|^2 \exp[-\mathrm{i}k(s-s')] - \int_0^\infty \mathrm{d}k\,|f(k)|^2 \exp[\mathrm{i}k(s-s')] \\ &\equiv \int_{-\infty}^\infty \mathrm{d}k\,\mathrm{sgn}(k)|f(k)|^2 \exp[-\mathrm{i}k(s-s')] \end{aligned} \tag{15}$$

where the symmetry $|f(k)|^2 = |f(-k)|^2$ has been assumed. This is to be compared with the commutation relation required in Streater's analysis, i.e.

$$[a(s), a^*(s')] = \int_0^\infty \mathrm{d}k\, |f(k)|^2 \exp[-\mathrm{i}k(s-s')]. \tag{16}$$

The commutation relation (15) is ued to compute the one for the solutions A_t and A_t^* of (13) and (13*), i.e.

$$A_t = A_0 \exp(-\mathrm{i}\omega - \gamma)t + \lambda \int_0^t \mathrm{d}s\, w(s) \exp[(-\mathrm{i}\omega-\gamma)(t-s)], \qquad A_t^* = \mathrm{HC},$$

so as to ensure the CCR condition

$$[A_t, A_t^*] = 1 \tag{17}$$

which is assumed to hold at the initial time $t=0$; $[A_0, A_0^*]=1$. This is completely analogous to what Streater did for his solutions of (1) and (1*) by using the commutation relation (16).

Our analysis, therfore, is quite parallel to Streater's treatment apart from the replacement of the integration $\int_0^\infty \mathrm{d}k|f(k)|^2 \ldots$ by the new integration $\int_{-\infty}^\infty \mathrm{d}k\, \mathrm{sgn}(k)|f(k)|^2 \ldots$, so that the integral equation for $|f(k)|^2$ may now be written as

$$\frac{\lambda^2}{2}\int_{-\infty}^\infty \mathrm{d}k[\mathrm{sgn}(k+\omega)|f(k+\omega)|^2 + \mathrm{sgn}(-k+\omega)|f(-k+\omega)|^2](\gamma^2+k^2)^{-1}\cos(kt)$$
$$= \mathrm{e}^{-\gamma t} \qquad t \geqslant 0. \tag{18}$$

This leads us, by virtue of the identity

$$\int_{-\infty}^\infty \mathrm{d}k \frac{\gamma}{\pi}(\gamma^2+k^2)^{-1}\cos(kt) = \mathrm{e}^{-\gamma t} \qquad t \geqslant 0$$

and the uniqueness of the inverse cosine transform, to

$$\mathrm{sgn}(k+\omega)\rho(k+\omega) + \mathrm{sgn}(-k+\omega)\rho(-k+\omega) = 2\gamma/\pi \tag{19}$$

where the same definition of the spectral density function $\rho(k)$ as before, equation (7), is made. We note that the sign function here ($\mathrm{sgn}(k)=+1$ for $k>0$ and -1 for $k<0$) is due to the basic commutation relation (6*a*) and is necessitated from our choice of the noise operator $w(t)$ in (14). Since any real number x is represented as $x = \mathrm{sgn}(x)|x|$, equation (19) may be rewritten as

$$\hat{\rho}(k+\omega) + \hat{\rho}(-k+\omega) = 2\gamma/\pi \tag{20}$$

where $\hat{\rho}(k)$, not necessarily positive, is related to the positive $\rho(k)$ as

$$\rho(k) = |\hat{\rho}(k)|. \tag{20a}$$

Thus, our functional equation to determine the quantum noise in the Langevin equation (1) is different from Streater's equation (8), though similar in form, in that the $\hat{\rho}(k)$ in (20) is neither restricted to positive values nor imposed to be identically zero on the negative k axis.

We expect, from a physical standpoint, that the function $\rho(k)$ should be factored out into a constant only depending on the parameters ω and γ times a universal function of k that represents the spectrum of the heat bath. Supposing that such a parameter dependence be absorbed into the coupling strength λ^2 in (7), we now wish to determine a possible form of the universal function that satisfies

$$\hat{\rho}(k+\omega)+\hat{\rho}(-k+\omega)=C(\omega,\gamma)=(2\gamma/\pi)\lambda^{-2}(\omega,\gamma) \tag{21}$$

where now $|\hat{\rho}(k)|$ $(=|f(k)|^2)$ is *independent of* ω *and* γ. Only a linear function $\hat{\rho}(k)=Ak+B$ satisfies this requirement, provided continuity is imposed! The proof of this fact is straightforward.

For $\omega=k$, (21) implies that $C(\omega,\gamma)=\hat{\rho}(2\omega)+\hat{\rho}(0)$ which leads to $\hat{\rho}(k+\omega)+\hat{\rho}(-k+\omega)=\hat{\rho}(2\omega)+\hat{\rho}(0)$. We let $g(k)\equiv\hat{\rho}(k)-\hat{\rho}(0)$, and set $a=k+\omega$ and $b=-k+\omega$: it follows that $g(a)+g(b)=g(a+b)$. Following a standard argument, we find that $2g(a)=g(2a)$ and $g(na)+g(a)=g((n+1)a)$, namely that $g(na)=ng(a)$. If $a=1/m$, then we have $g(n/m)=ng(1/m)$. Finally, we add the relation for $n=m$, namely $g(1)=mg(1/m)$, which leads to $g(n/m)=(n/m)g(1)$. By the assumption of continuity, as $n/m\to k$, an arbitrary positive real, we find that $g(k)=kg(1)$. For negative k one chooses $a=-1/m$ in the above argument leading to $g(-k)=kg(-1)$. Since $g(k)+g(-k)=g(0)=0$ it follows that $g(-1)=-g(1)$, and thus $g(k)=kg(1)$ holds for all real k. Re-expressing this result in terms of $\hat{\rho}$ we find $\hat{\rho}(k)=kg(1)+\hat{\rho}(0)$, as was to be shown.

Having the possible form of $\hat{\rho}(k)$ so determined, we further specify it by imposing

$$\hat{\rho}(0)=0 \tag{22}$$

which is necessary for the regularity of the equilibrium state (10) with the Planck distribution (11) in the limit $k\to 0$. The determination of $\hat{\rho}(k)$ and hence $\rho(k)$ is now unique, i.e.

$$\rho(k)=\lambda^2|f(k)|^2=(\gamma/\pi\omega)|k|, \tag{23}$$

implying that the universal function to represent the spectral density of the boson heat bath must be the linear function $|k|$.

It should be noted that the modification of the quantum noise from $a(t)$ to $w(t)$ i.e. from equation (1) to (13) (also from equation (1*) to (13*)) affects a different part of Streater's results than the spectrum: the gauge invariance no longer holds so that $\langle A_t A_{t'}\rangle\neq 0$. On the other hand, the second requirement (ii) of the KMS stationary state is still satisfied even with the sbove alteration, as can be seen from

$$\langle A^*_\infty A_\infty(\tau)\rangle_\beta=\frac{\gamma}{\pi\omega}\int_{-\infty}^{\infty}\mathrm{d}k[\theta(k)kn(k)+\theta(-k)|k|(n|k|)+1)]\frac{\mathrm{e}^{-ik\tau}}{(k-\omega)^2+\gamma^2} \tag{24a}$$

and

$$\langle A_\infty(\tau)A^*_\infty\rangle_\beta=\frac{\gamma}{\pi\omega}\int_{-\infty}^{\infty}\mathrm{d}k[\theta(k)k(n(k)+1)+\theta(-k)|k|n(|k|)]\frac{\mathrm{e}^{-ik\tau}}{(k-\omega)^2+\gamma^2} \tag{24b}$$

where $\theta(k)$ is the Heaviside function and

$$(n(|k|)+1)\,\mathrm{e}^{-\beta\hbar|k|}=n(|k|), \tag{25}$$

and hence $\langle A^*_\infty A_\infty(\tau+\mathrm{i}\beta)\rangle_\beta=\langle A_\infty(\tau)A^*_\infty\rangle_\beta$.

Equations (24a) and (24b) are combined to give

$$\langle\tfrac{1}{2}(A^*_\infty A_\infty(\tau)+A_\infty(\tau)A^*_\infty)\rangle_\beta$$
$$=\frac{\beta^{-1}}{\hbar\omega}\int_{-\infty}^{\infty}\mathrm{d}k\,\mu(k)\frac{\gamma}{2\pi}\left(\frac{\mathrm{e}^{-ik\tau}}{(k-\omega)^2+\gamma^2}+\frac{\mathrm{e}^{ik\tau}}{(k+\omega)^2+\gamma^2}\right) \tag{26}$$

where

$$\mu(k)=\tfrac{1}{2}\beta\hbar k \coth \tfrac{1}{2}\beta\hbar k, \tag{26a}$$

and

$$[A_\infty(\tau), A_\infty^*]=\frac{1}{\omega}\int_{-\infty}^{\infty} \mathrm{d}k\, k\frac{\gamma}{2\pi}\left(\frac{\mathrm{e}^{-\mathrm{i}k\tau}}{(k-\omega)^2+\gamma^2}-\frac{\mathrm{e}^{\mathrm{i}k\tau}}{(k+\omega)^2+\gamma^2}\right). \tag{27}$$

In particular for $\tau=0$, $[A_\infty, A_\infty^*]=1$ as must be the case, and

$$X_\infty \equiv \langle\tfrac{1}{2}(A_\infty^* A_\infty + A_\infty A_\infty^*)\rangle_\beta$$
$$=\frac{\beta^{-1}}{\hbar\omega}\int_{-\infty}^{\infty} \mathrm{d}k\,\mu(k)\frac{1}{2\pi}\left(\frac{\gamma}{(k-\omega)^2+\gamma^2}+\frac{\gamma}{(k+\omega)^2+\gamma^2}\right) \tag{28}$$

which may be considered as the quantum analogue of the fluctuation dissipation theorem implied in (3), modulo zero point flcutuations. One can show in fact the approach of the operator X_t to X_∞ from the Langevin equation

$$\mathrm{d}X_t/\mathrm{d}t=-2\gamma(X_t-X_\infty)+F(t), \qquad \langle F(t)\rangle_\beta \xrightarrow{t\to\infty} 0. \tag{29}$$

Finally, the two results (26) and (27) indicate that the stationary quantum stochastic process so established as solutions to equations (13) and (13*) provides a good example of the operator-valued Gaussian process first formulated by Lewis and Thomas (1975), because of the structure $[A_\infty(\tau), A_\infty^*]=\mathrm{i}\hbar\beta\psi'(\tau)$ where $\psi(\tau)$ is the correlation function in (26) with $\mu(k)$ being replaced by the classical limit (1). It is also noteworthy that the universal noise operator $w(t)$, renormalised such that $[w(t), w(0)]=\mathrm{i}\hbar\beta\delta'(t)$ can be identified with the recently introduced notion of the quantum Gaussian (white) noise (Nakazawa 1985). From the above Lewis–Thomas point of view, the correlation function $\psi(\tau)$ is equal to $\delta(\tau)$ for which the power spectrum is just 'white' and the quantum effect is represented by the universal function $\mu(k)$ given by (26a).

Two of us (JRK and ML) would like to acknowledge the Japan Society for Promotion of Science for financial support.

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